The (2+1) and (3+1)-Dimensional CBS Equations: Multiple Soliton Solutions and Multiple Singular Soliton Solutions

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In this work, the generalized (2+1) and (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equations are studied. We employ the Cole-Hopf transformation and the Hirota bilinear method to derive multiple-soliton solutions and multiple singular soliton solutions for these equations. The necessary conditions for complete integrability of each equation are derived.

Key words: Calogero-Bogoyavlenskii-Schiff Equations; Hirota's Method; Multiple Solitons; Multiple Singular Solitons.

1. Introduction

In this work, we will study the generalized (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equations [1 – 9]

$$v_t + \Phi(v)v_y = 0, \quad \Phi(v) = \partial_x^2 + av + bv_x\partial_x^{-1}, \tag{1}$$

or equivalently

$$v_t + v_{xxy} + avv_y + bv_x\partial_x^{-1}v_y = 0, \tag{2}$$

where

$$\partial_x^{-1}f = \int f\,\mathrm{d}x. \tag{3}$$

Moreover, we will also study the generalized (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equations

$$\begin{aligned} &v_t + \Phi(v)v_y + \Phi_1(v)v_z = 0,\\ &\Phi(v) = \partial_x^2 + av + bv_x\partial_x^{-1},\\ &\Phi_1(v) = \partial_x^2 + cv + dv_x\partial_x^{-1}, \end{aligned} \tag{4}$$

or equivalently

$$\begin{aligned} &v_t + avv_y + cvv_z + bv_x\partial_x^{-1}v_y\\ &\quad + dv_x\partial_x^{-1}v_z + v_{xxy} + v_{xxz} = 0, \end{aligned} \tag{5}$$

where a, b, c, and d are parameters. Using a dimensional reduction $\partial_z = \partial_y = \partial_x$, (2) and (5) will be reduced to the standard Korteweg-de Vries (KdV) equation.

The (2+1)-dimensional CBS equation (2) can be written in the potential form

$$u_{xt} + au_xu_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \tag{6}$$

obtained upon using the potential $v = u_x$. Similarly, the (3+1)-dimensional CBS equation (5) can be written in the potential form

$$\begin{aligned} &u_{xt} + au_xu_{xy} + bu_{xx}u_y + cu_xu_{xz}\\ &\quad + du_{xx}u_z + u_{xxxy} + u_{xxxz} = 0, \end{aligned} \tag{7}$$

obtained upon using the potential $v = u_x$. The CBS equation was first constructed by Bogoyavlenskii and Schiff in different ways [2 – 4]. Bogoyavlenskii used the modified Lax formalism, whereas Schiff derived the same equation by reducing the self-dual Yang-Mills equation [1 – 9].

For completely integrable evolution equations, three powerful methods, namely the inverse scattering method, the Bäcklund transformation method, and the Hirota bilinear method [10 – 13] were thoroughly used to derive the multiple-soliton solutions of these equations. Other useful methods are used in [14 – 19]. The Hirota's bilinear method is rather heuristic and possesses significant features that make it practical for the determination of multiple-soliton solutions [19 – 24] for a wide class of nonlinear evolution equations in a direct method. Moreover, the tanh method was used to determine single-soliton solutions. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

The objectives of this work are twofold. First, we seek to extend our work in [1] to establish multiple soliton solutions and multiple singular soliton solutions of distinct physical structures for the generalized (2+1) and (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equations. The Cole-Hopf transformation and Hirota's bilinear sense will be used to achieve the first goal. The second goal is to show that the complete integrability of these equations is subjected to necessary conditions related to the parameters a, b, c, and d.

2. The Hirota Method

In what follows we briefly highlight the main features of Hirota's bilinear method that will be used in this work. We first substitute

$$u(x,y,zt) = \mathrm{e}^{kx+my+rz-\omega t} \tag{8}$$

into the linear terms of any equation under discussion to determine the relation between k, m, r, and ω. We then substitute the Cole-Hopf transformation

$$u = R(\ln f)_x = R\frac{f_x}{f} \tag{9}$$

into the equation under discussion, where the auxiliary function f, for the single soliton solution, is given by

$$f(x,y,z,t) = 1 + f_1(x,y,z,t) = 1 + \mathrm{e}^{\theta_1}. \tag{10}$$

The steps of the Hirota method, summarized in [1], are as follows:

(i) For the relation between k_i, m_i, r_i, and ω_i, we use

$$u(x,y,z,t) = \mathrm{e}^{\theta_i}, \quad \theta_i = k_i x + m_i y + r_i z - \omega_i t. \tag{11}$$

(ii) For single soliton, we use

$$f = 1 + \mathrm{e}^{\theta_1} \tag{12}$$

to determine R.

(iii) For two-soliton solutions, we use

$$f = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \tag{13}$$

to determine the phase shift coefficient a_{12}, which can be generalized for a_{ij}, $1 \le i < j \le 3$.

(iv) For three-soliton solutions, we use

$$\begin{aligned} f = 1 &+ \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \\ &+ a_{23}\mathrm{e}^{\theta_2+\theta_3} + a_{13}\mathrm{e}^{\theta_1+\theta_3} + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{14}$$

to determine b_{123}. It is formally proved that if $b_{123} = a_{12}a_{23}a_{13}$, then the equation gives rise to three-soliton solutions. The determination of three-soliton solutions confirms the fact that N-soliton solutions exist for any order, and hence, the examined equation is completely integrable.

However, for the multiple singular soliton solutions [14–18], we use the following steps:

(i) For dispersion relation, we use

$$u(x,y,z,t) = \mathrm{e}^{\theta_i}, \ \theta_i = k_i x + m_i y + r_i z - \omega_i t. \tag{15}$$

(ii) For single singular soliton, we use

$$f(x,y,z,t) = 1 - \mathrm{e}^{\theta_1}. \tag{16}$$

(iii) For two singular soliton solutions, we use

$$f(x,y,z,t) = 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2}. \tag{17}$$

(iv) For three singular soliton solutions, we use

$$\begin{aligned} f(x,y,z,t) = {} & 1 - \mathrm{e}^{\theta_1} - \mathrm{e}^{\theta_2} - \mathrm{e}^{\theta_3} \\ & + a_{12}\mathrm{e}^{\theta_1+\theta_2} + a_{23}\mathrm{e}^{\theta_2+\theta_3} \\ & + a_{13}\mathrm{e}^{\theta_1+\theta_3} + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{18}$$

3. The (2+1)-Dimensional CBS Equation

The potential form of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation [1–9] is given by

$$u_{xt} + u_{xxxy} + au_x u_{xy} + bu_{xx}u_y = 0, \tag{19}$$

where $u = u(x,y,t)$. The derivation of the potential form is given above in (1)–(7). Our approach depends mainly on the Cole-Hopf transformation and Hirota's direct method as summarized before.

We first substitute

$$u(x,y,t) = \mathrm{e}^{k_i x + m_i y - \omega_i t} \tag{20}$$

into the linear terms of (19) to find the relation

$$\omega_i = k_i^2 m_i, \ i = 1, 2, \cdots N \tag{21}$$

and hence θ_i becomes

$$\theta_i = \mathrm{e}^{k_i x + m_i y - k_i^2 m_i t}. \tag{22}$$

To determine R, we substitute the Cole-Hopf transformation

$$u(x,y,t) = R(\ln f)_x, \tag{23}$$

where

$$f(x,y,t) = 1 + f_1(x,y,t) = 1 + \mathrm{e}^{\theta_1}, \tag{24}$$

and insert this into (19) to find that

$$R = \frac{12}{a+b}. \tag{25}$$

This in turn defines the solution $u(x,y,t)$ by

$$u(x,y,t) = \frac{12}{a+b}(\ln f)_x. \tag{26}$$

3.1. Multiple-Soliton Solutions for $a = 4, b = 2$

We found that the complete integrability for the (2+1)-dimensional CBS equation for the case $a = 4$ and $b = 2$ is justified for distinct coefficients, $k_i \neq m_i$, of the spatial variables x and y. Using (24) and (26) for $R = 2$, the single-soliton solution is given by

$$u(x,y,t) = \frac{2k_1 \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}{1 + \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}. \tag{27}$$

Recall that $v(x,y,t) = u_x(x,y,t)$. This gives the single-soliton solution of the CBS equation by

$$v(x,y,t) = \frac{2k_1^2 \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}{(1 + \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t})^2}. \tag{28}$$

For the two-soliton solutions we substitute

$$f(x,y,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + a_{12}\mathrm{e}^{\theta_1+\theta_2} \tag{29}$$

into (19) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{30}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},\ 1 \le i < j \le N. \tag{31}$$

Notice that the phase shifts a_{ij} do not depend on m_i. This in turn gives

$$\begin{aligned} f(x,y,t) &= 1 + \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t} + \mathrm{e}^{k_2 x + m_2 y - k_2^2 m_2 t} \\ &+ \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\mathrm{e}^{(k_1+k_2)x + (m_1+m_2)y - (k_1^2 m_1 + k_2^2 m_2)t}. \end{aligned} \tag{32}$$

To determine the two-soliton solutions explicitly, we substitute (32) into the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Recall again that $v(x,y,t) = u_x(x,y,t)$.

It is interesting to point out that the CBS equation (19) does not show any resonant phenomenon [10] because the phase shift term a_{12} in (30) cannot be 0 or ∞ for $|k_1| \neq |k_2|$. It is well known that a two-soliton solution [10] can degenerate into a resonant triad under the conditions

$$a_{12} = 0 \text{ or } (a_{12})^{-1} = 0 \text{ for } |k_1| \neq |k_2|. \tag{33}$$

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) &= 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \mathrm{e}^{\theta_3} \\ &+ a_{12}\mathrm{e}^{\theta_1+\theta_2} + a_{23}\mathrm{e}^{\theta_2+\theta_3} \\ &+ a_{13}\mathrm{e}^{\theta_1+\theta_3} + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{34}$$

into (19) and, solving for b_{123}, we find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{35}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$. The higher level soliton solutions, for $n \ge 4$ can be obtained in a parallel manner. This confirms that the CBS equation is completely integrable and admits multiple-soliton solutions of any order.

3.2. Multiple Singular Soliton Solutions for $a = 4$, $b = 2$

We found that the multiple singular soliton solutions for the (2+1)-dimensional CBS equation for the case $a = 4$ and $b = 2$ exist for distinct coefficients, $k_i \neq m_i$, of the spatial variables x and y. In this case $R = 2$ as derived before. As stated above, the auxiliary function for the singular soliton solutions is given by

$$f(x,y,t) = 1 - f_1(x,y,t) = 1 - \mathrm{e}^{\theta_1}. \tag{36}$$

Using (36) and (26) for $R = 2$, the single singular soliton solution is given by

$$u(x,y,t) = -\frac{2k_1 \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}{1 - \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}. \tag{37}$$

Recall that $v = u_x$. This means that the singular soliton solution is given by

$$v(x,y,t) = -\frac{2k_1^2 \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t}}{(1 - \mathrm{e}^{k_1 x + m_1 y - k_1^2 m_1 t})^2}. \tag{38}$$

For the two singular soliton solutions we substitute

$$f(x,y,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{39}$$

into (19) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{40}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ 1 \le i < j \le N. \tag{41}$$

Notice that the phase shifts a_{ij} do not depend on m_i. This in turn gives

$$f(x,y,t) = 1 - e^{k_1x+m_1y-k_1^2m_1t} - e^{k_2x+m_2y-k_2^2m_2t} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x+(m_1+m_2)y-(k_1^2m_1+k_2^2m_2)t}. \tag{42}$$

To determine the two-soliton solutions explicitly, we substitute (42) into the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$.

Similarly, to determine the three singular soliton solutions, we set

$$\begin{aligned} f(x,y,t) = 1 &- e^{\theta_1} - e^{\theta_2} - e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{43}$$

into (19) and, solving for b_{123}, we find that

$$b_{123} = -a_{12}a_{13}a_{23}. \tag{44}$$

To determine the three singular soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$. The higher level singular soliton solutions for $n \ge 4$ can be obtained in a parallel manner. This confirms that the CBS equation admits multiple-soliton solutions and multiple singular soliton solutions of any order.

3.3. Multiple-Soliton Solutions for Arbitrary a and b

In this part we examine the complete integrability for a generalized (2+1)-dimensional equation

$$u_{xt} + u_{xxxy} + au_xu_{xy} + bu_{xx}u_y = 0, \tag{45}$$

where a and b can be any arbitrary constants.

We found that the complete integrability for the (2+1)-dimensional CBS equation (45) for any arbitrary constants a and b is justified only if $m_i = k_i$, of the spatial variables x and y. In this case $R = \frac{12}{a+b}$ as derived before. Using (24) and (26) for $R = \frac{12}{a+b}$, the single soliton solution is given by

$$u(x,y,t) = \frac{12k_1e^{k_1x+k_1y-k_1^3t}}{(a+b)(1+e^{k_1x+k_1y-k_1^3t})}. \tag{46}$$

Noting that $v(x,y,t) = u_x(x,y,t)$, therefore we obtain

$$v(x,y,t) = \frac{12k_1^2e^{k_1x+k_1y-k_1^3t}}{(a+b)(1+e^{k_1x+k_1y-k_1^3t})^2}. \tag{47}$$

For the two-soliton solutions we substitute

$$f(x,y,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{48}$$

into (45) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{49}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ 1 \le i < j \le N. \tag{50}$$

Notice that the phase shifts a_{ij} remain the same as in the previous case. This in turn gives

$$f(x,y,t) = 1 + e^{k_1x+k_1y-k_1^3t} + e^{k_2x+k_2y-k_2^3t} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x+(k_1+k_2)y-(k_1^3+k_2^3)t}. \tag{51}$$

To determine the two-soliton solutions explicitly, we substitute (51) into the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Note that $v(x,y,t) = u_x(x,y,t)$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,t) = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{52}$$

into (45) and, solving for b_{123}, we find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{53}$$

To determine the three-soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Note that $v(x,y,t) = u_x(x,y,t)$. The higher level soliton solutions for $n \geq 4$ can be obtained in a parallel manner. This confirms that the generalized CBS equation (45) is completely integrable and admits multiple-soliton solutions of any order for any arbitrary constants a and b, provided that the coefficients of the spatial variables x and y are identical.

3.4. Multiple Singular Soliton Solutions for Arbitrary a and b

We found that the multiple singular soliton solutions for the (2+1)-dimensional CBS equation (45) for arbitrary a and b exist only if the coefficients k_i and m_i of the spatial variables x and y are identical. In this case $R = \frac{12}{a+b}$ as derived before. As stated above, the auxiliary function for the singular soliton solutions is given by

$$f(x,y,t) = 1 + f_1(x,y,t) = 1 - e^{\theta_1}. \tag{54}$$

Using (54) and (26) for $R = \frac{12}{a+b}$, the single singular soliton solution is given by

$$u(x,y,t) = -\frac{12k_1 e^{k_1x+k_1y-k_1^3t}}{(a+b)(1-e^{k_1x+k_1y-k_1^3t})}. \tag{55}$$

Noting that $v(x,y,t) = u_x(x,y,t)$ gives

$$v(x,y,t) = -\frac{12k_1^2 e^{k_1x+k_1y-k_1^3t}}{(a+b)(1-e^{k_1x+k_1y-k_1^3t})^2}. \tag{56}$$

For the two singular soliton solutions we substitute

$$f(x,y,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{57}$$

into (45) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{58}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2},\ 1 \leq i < j \leq N. \tag{59}$$

Notice that the phase shifts a_{ij} is the same as obtained before. This in turn gives

$$\begin{aligned} f(x,y,t) = 1 &- e^{k_1x+k_1y-k_1^3t} - e^{k_2x+k_2y-k_2^3t} \\ &+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2} e^{(k_1+k_2)x+(k_1+k_2)y-(k_1^3+k_2^3)t}. \end{aligned} \tag{60}$$

To determine the two-soliton solutions explicitly, we substitute (60) into the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$.

Similarly, to determine the three singular soliton solutions, we set

$$\begin{aligned} f(x,y,t) = 1 &- e^{\theta_1} - e^{\theta_2} - e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{61}$$

into (45) and, solving for b_{123}, we find that

$$b_{123} = -a_{12}a_{13}a_{23}. \tag{62}$$

To determine the three singular soliton solutions explicitly, we substitute the last result for $f(x,y,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,t))_x$. The higher level singular soliton solutions for $n \geq 4$ can be obtained in a parallel manner. This confirms that the CBS equation admits multiple-soliton solutions and multiple singular soliton solutions of any order.

4. The (3+1)-Dimensional CBS Equation

The potential form of the (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation is given by

$$\begin{aligned} &u_{xt} + au_xu_{xy} + bu_{xx}u_y + cu_xu_{xz} + du_{xx}u_z \\ &\quad + u_{xxxy} + u_{xxxz} = 0, \end{aligned} \tag{63}$$

where $u = u(x,y,z,t)$. The derivation of the potential form is given above in (1)–(7). Our approach depends mainly on the Cole-Hopf transformation and the Hirota direct method as summarized before.

We first substitute

$$u(x,y,z,t) = e^{k_ix+m_iy+r_iz-\omega_it} \tag{64}$$

into the linear terms of (63) to find the relation

$$\omega_i = k_i^2m_i + k_i^2r_i,\ i = 1,2,\cdots N, \tag{65}$$

and hence θ_i becomes

$$\theta_i = e^{k_ix+m_iy+r_iz-k_i^2(m_i+r_i)t}. \tag{66}$$

To determine R, we substitute the Cole-Hopf transformation

$$u(x,y,z,t) = R(\ln f)_x, \tag{67}$$

where

$$f(x,y,z,t) = 1 + f_1(x,y,z,t) = 1 + e^{\theta_1}, \quad (68)$$

and insert this into (63) to find that

$$R = \frac{24}{a+b+c+d}. \quad (69)$$

This in turn defines the solution $u(x,y,z,t)$ by

$$u(x,y,z,t) = \frac{24}{a+b+c+d}(\ln f)_x. \quad (70)$$

4.1. Multiple-Soliton Solutions for $a = c = 4$, $b = d = 2$

We found that the complete integrability for the (3+1)-dimensional CBS equation (63) for the case $a = c = 4$ and $b = d = 2$ is justified for distinct coefficients, $r_i = m_i, k_i \neq m_i$. In this case $R = 2$, and

$$\theta_i = e^{k_i x + m_i y + m_i z - 2k_i^2 m_i t}. \quad (71)$$

Using (68) and (70) for $R = 2$, the single soliton solution is given by

$$u(x,y,z,t) = \frac{2k_1 e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}{1 + e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}. \quad (72)$$

Recall that $v(x,y,z,t) = u_x(x,y,z,t)$. This gives the single soliton solution of the CBS equation by

$$v(x,y,t) = \frac{2k_1^2 e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}{(1 + e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t})^2}. \quad (73)$$

For the two-soliton solutions we substitute

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2} \quad (74)$$

into (63) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)(k_1^2 m_2 + 2k_1 k_2 (m_1 - m_2) - k_2^2 m_1)}{(k_1 + k_2)(k_1^2 m_2 + 2k_1 k_2 (m_1 + m_2) + k_2^2 m_1)}, \quad (75)$$

and this can be generalized to

$$a_{ij} = \frac{(k_i - k_j)(k_i^2 m_j + 2k_i k_j (m_i - m_j) - k_j^2 m_i)}{(k_i + k_j)(k_i^2 m_j + 2k_i k_j (m_i + m_j) + k_j^2 m_i)}, \quad (76)$$
$$1 \leq i < j \leq N.$$

Notice that the phase shifts a_{ij} depend on both coefficients k_i and m_i. This in turn gives

$$\begin{aligned} f(x,y,z,t) = \\ 1 &+ e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t} + e^{k_2 x + m_2 y + m_2 z - 2k_2^2 m_2 t} \\ &+ \frac{(k_1 - k_2)(k_1^2 m_2 + 2k_1 k_2 (m_1 - m_2) - k_2^2 m_1)}{(k_1 + k_2)(k_1^2 m_2 + 2k_1 k_2 (m_1 + m_2) + k_2^2 m_1)} \\ &\cdot e^{(k_1+k_2)x + (m_1+m_2)y + (m_1+m_2)z - (2k_1^2 m_1 + 2k_2^2 m_2)t}. \end{aligned} \quad (77)$$

To determine the two-soliton solutions explicitly, we substitute (77) into the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall again that $v(x,y,z,t) = u_x(x,y,z,t)$.

Similarly, to determine the three-soliton solutions, we set

$$\begin{aligned} f(x,y,z,t) = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \\ &+ a_{12} e^{\theta_1 + \theta_2} + a_{23} e^{\theta_2 + \theta_3} \\ &+ a_{13} e^{\theta_1 + \theta_3} + b_{123} e^{\theta_1 + \theta_2 + \theta_3} \end{aligned} \quad (78)$$

into (63) and, solving for b_{123}, we find that

$$b_{123} = a_{12} a_{13} a_{23}. \quad (79)$$

To determine the three-soliton solutions, we substitute the last result for $f(x,y,z,t)$ in the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,z,t) = u_x(x,y,z,t)$. The higher level soliton solutions for $n \geq 4$ can be obtained in a parallel manner. This confirms that the CBS equation is completely integrable and admits multiple-soliton solutions of any order.

4.2. Multiple Singular Soliton Solutions for $a = c = 4, b = d = 2$

To determine singular soliton solutions, the auxiliary function for the singular soliton solutions is given by

$$f(x,y,z,t) = 1 + f_1(x,y,z,t) = 1 - e^{\theta_1}. \quad (80)$$

Using (80) and (70) for $R = 2$, the single singular soliton solution is given by

$$u(x,y,z,t) = -\frac{2k_1 e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}{1 - e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}. \quad (81)$$

Recall that $v = u_x$. This means that the singular soliton solution is given by

$$v(x,y,z,t) = -\frac{2k_1^2 e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t}}{(1 - e^{k_1 x + m_1 y + m_1 z - 2k_1^2 m_1 t})^2}. \quad (82)$$

For the two singular soliton solutions we substitute

$$f(x,y,z,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{83}$$

into (63) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1-k_2)(k_1^2m_2 + 2k_1k_2(m_1-m_2) - k_2^2m_1)}{(k_1+k_2)(k_1^2m_2 + 2k_1k_2(m_1+m_2) + k_2^2m_1)}, \tag{84}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)(k_i^2m_j + 2k_ik_j(m_i-m_j) - k_j^2m_i)}{(k_i+k_j)(k_i^2m_j + 2k_ik_j(m_i+m_j) + k_j^2m_i)}, \quad 1 \le i < j \le N. \tag{85}$$

Notice that the phase shifts a_{ij} depend on the coefficients k_i and m_i. This in turn gives

$$\begin{aligned} f(x,y,z,t) &= \\ 1 &- e^{k_1x+m_1y+m_1z-2k_1^2m_1t} - e^{k_2x+m_2y+m_2z-2k_2^2m_2t} \\ &+ \frac{(k_1-k_2)(k_1^2m_2 + 2k_1k_2(m_1-m_2) - k_2^2m_1)}{(k_1+k_2)(k_1^2m_2 + 2k_1k_2(m_1+m_2) + k_2^2m_1)} \\ &\cdot e^{(k_1+k_2)x+(m_1+m_2)y+(m_1+m_2)z-(2k_1^2m_1+2k_2^2m_2)t}. \end{aligned} \tag{86}$$

To determine the two-soliton solutions explicitly, we substitute (86) into the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$.

Similarly, to determine the three singular soliton solutions, we substitute

$$\begin{aligned} f(x,y,z,t) = 1 &- e^{\theta_1} - e^{\theta_2} - e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{87}$$

into (63) and, solving for b_{123}, we find that

$$b_{123} = -a_{12}a_{13}a_{23}. \tag{88}$$

To determine the three singular soliton solutions explicitly, we substitute the last result for $f(x,y,z,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$. The higher level singular soliton solutions for $n \ge 4$ can be obtained in a parallel manner. This confirms that the CBS equation admits multiple-soliton solutions and multiple singular soliton solutions of any order.

4.3. Multiple-Soliton Solutions for Arbitrary a,b,c, and d

We found that the complete integrability for the (3+1)-dimensional CBS equation (63) for arbitrary a,b,c, and d is justified only if $m_i = k_i$. In this case $R = \frac{24}{a+b+c+d}$ and

$$\theta_i = e^{k_ix+k_iy+k_iz-2k_i^3t}. \tag{89}$$

Using (68) and (70) for $R = \frac{24}{a+b+c+d}$, the single-soliton solution is given by

$$u(x,y,z,t) = \frac{24k_1e^{k_1x+k_1y+k_1z-2k_1^3t}}{(a+b+c+d)(1+e^{k_1x+k_1y+k_1z-2k_1^3t})}. \tag{90}$$

Recall that $v(x,y,z,t) = u_x(x,y,z,t)$. This gives the single-soliton solution of the CBS equation by

$$v(x,y,t) = \frac{24k_1^2e^{k_1x+k_1y+k_1z-2k_1^3t}}{(a+b+c+d)(1+e^{k_1x+k_1y+k_1z-2k_1^3t})^2}. \tag{91}$$

For the two-soliton solutions we substitute

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{92}$$

into (63) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{93}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j^2)}{(k_i+k_j^2)}, \; 1 \le i < j \le N. \tag{94}$$

This in turn gives

$$\begin{aligned} f(x,y,z,t) &= \\ 1 &+ e^{k_1x+k_1y+k_1z-2k_1^3t} + e^{k_2x+k_2y+k_2z-2k_2^3t} \\ &+ \frac{(k_1-k_2)^2}{(k_1+k_2)^2}\, e^{(k_1+k_2)x+(k_1+k_2)y+(k_1+k_2)z-2(k_1^3+k_2^3)t}. \end{aligned} \tag{95}$$

To determine the two-soliton solutions explicitly, we substitute (95) into the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall again that $v(x,y,z,t) = u_x(x,y,z,t)$.

Similarly, to determine the three-soliton solutions, we substitute

$$\begin{aligned} f(x,y,z,t) = 1 &+ e^{\theta_1} + e^{\theta_2} + e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{96}$$

into (63) and, solving for b_{123}, we find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{97}$$

To determine the three-soliton solutions, we substitute the last result for $f(x,y,z,t)$ in the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,z,t) = u_x(x,y,z,t)$. The higher level soliton solutions for $n \geq 4$ can be obtained in a parallel manner. This confirms that the (3+1)-dimensional CBS equation is completely integrable and admits multiple-soliton solutions of any order for arbitrary values of a and b.

4.4. Multiple Singular Soliton Solutions for Arbitrary a, b, c, and d

To determine singular soliton solutions, the auxiliary function for the singular soliton solutions is given by

$$f(x,y,z,t) = 1 + f_1(x,y,z,t) = 1 - e^{\theta_1}. \tag{98}$$

Proceeding as before, and using $R = \frac{24}{a+b+c+d}$, the single singular soliton solution is given by

$$u(x,y,z,t) = -\frac{24k_1 e^{k_1 x + k_1 y + k_1 z - 2k_1^3 t}}{(a+b+c+d)(1 - e^{k_1 x + k_1 y + k_1 z - 2k_1^3 t})}. \tag{99}$$

Recall that $v = u_x$. This means that the singular soliton solution is given by

$$v(x,y,z,t) = -\frac{24k_1^2 e^{k_1 x + k_1 y + k_1 z - 2k_1^3 t}}{(a+b+c+d)(1 - e^{k_1 x + k_1 y + k_1 z - 2k_1^3 t})^2}. \tag{100}$$

For the two singular soliton solutions we substitute

$$f(x,y,z,t) = 1 - e^{\theta_1} - e^{\theta_2} + a_{12}e^{\theta_1+\theta_2} \tag{101}$$

into (63) and, solving for the phase shift a_{12}, we find

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{102}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},\ 1 \leq i < j \leq N. \tag{103}$$

This in turn gives

$$\begin{aligned} f(x,y,z,t) &= 1 - e^{k_1 x + k_1 y + k_1 z - 2k_1^3 t} - e^{k_2 x + k_2 y + k_2 z - 2k_2^3 t} \\ &+ \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{(k_1+k_2)x + (k_1+k_2)y + (k_1+k_2)z - 1(2k_1^3 + 2k_2^3)t}. \end{aligned} \tag{104}$$

To determine the two singular soliton solutions explicitly, we substitute (104) into the formula $u(x,y,z,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$.

Similarly, to determine the three singular soliton solutions, we substitute

$$\begin{aligned} f(x,y,z,t) = 1 &- e^{\theta_1} - e^{\theta_2} - e^{\theta_3} \\ &+ a_{12}e^{\theta_1+\theta_2} + a_{23}e^{\theta_2+\theta_3} \\ &+ a_{13}e^{\theta_1+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{105}$$

into (63), and solve for b_{123}, we find that

$$b_{123} = -a_{12}a_{13}a_{23}. \tag{106}$$

To determine the three singular soliton solutions explicitly, we substitute the last result for $f(x,y,z,t)$ in the formula $u(x,y,t) = 2(\ln f(x,y,z,t))_x$. Recall that $v(x,y,t) = u_x(x,y,t)$. The higher level singular soliton solutions for $n \geq 4$ can be obtained in a parallel manner. This confirms that the (3+1)-dimensional CBS equation admits multiple-soliton solutions and multiple singular soliton solutions of any order.

5. Concluding Remarks

The (2+1)-dimensional and the (3+1)-dimensional Calogero-Bogoyavlenskii-Schiff equations are investigated for multiple-soliton solutions and multiple singular soliton solutions. The Cole-Hopf transformation and the Hirota bilinear method are used to formally derive these solutions. The solutions were obtained for the well-known forms of the CBS equations with fixed values for the parameters a, b and a, b, c, and d. The generalized forms were also studied and solutions were obtained for arbitrary values of the constants. The analysis highlights the power of Hirota's method compared to other existing techniques.

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